Name:	
Class:	

JAMES RUSE A H S
GESTA NON VERBA

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2012

MATHEMATICS EXTENSION 2

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 3 hours.
- Write in black or blue pen.
- Board approved calculators & templates may be used
- A Standard integral Sheet is provided.
- In every question, show all necessary working
- Marks may not be awarded for careless or badly arranged working.

Total Marks 100

Section I: 10 marks

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: 90 Marks

- Attempt Question 11 - 16
- Answer on blank paper unless otherwise instructed. Start a new page for each new question.
- Allow about 2 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labeled Question 11, Question 12, etc. Each question must show your Candidate Number.

Section I

10 Marks
Attempt Question 1 – 10.
Allow approximately 15 minutes for this section.

Use the multiple choice answer sheet below to record your answers to Question 1 – 10.

Select the alternative: A, B, C or D that best answers the question.

Colour in the response oval completely.

Sample:

$2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, draw a cross through the incorrect answer and colour in the new answer

ie A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word "*correct*" and draw an arrow as follows:

correct

A ● B ⊗ C ○ D ○

Trial HSC Examination
Mathematics Extension 2, 2012

Multiple Choice Answer Sheet

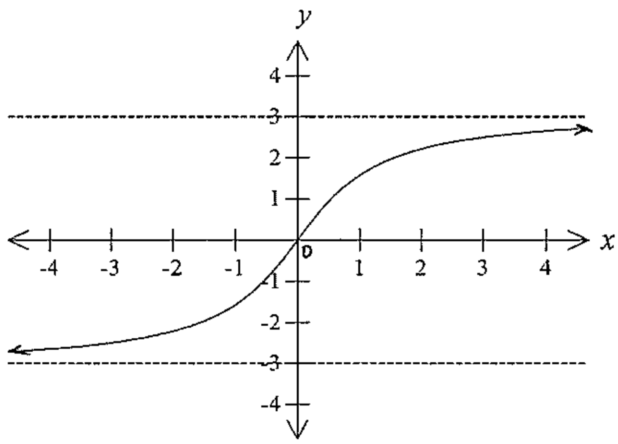
Student id number: | | | | |

Completely colour in the response oval representing the most correct answer.

1	A	○	B	○	C	○	D	○
2	A	○	B	○	C	○	D	○
3	A	○	B	○	C	○	D	○
4	A	○	B	○	C	○	D	○
5	A	○	B	○	C	○	D	○
6	A	○	B	○	C	○	D	○
7	A	○	B	○	C	○	D	○
8	A	○	B	○	C	○	D	○
9	A	○	B	○	C	○	D	○
10	A	○	B	○	C	○	D	○

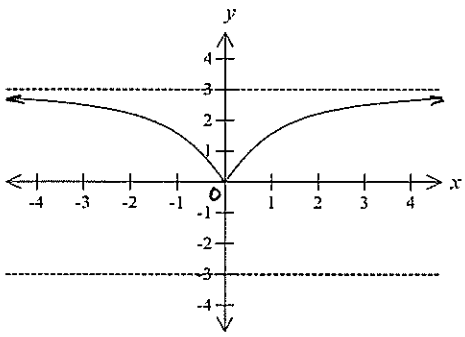
Mark: /10

1. The diagram shows the graph of the function $y = f(x)$.

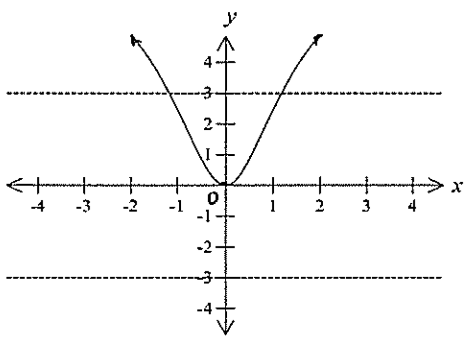


Which of the following is the graph of $y = \sqrt{f(x)}$?

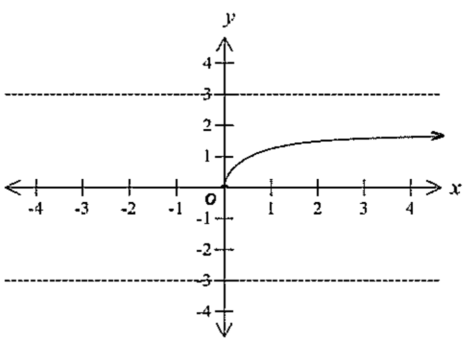
(A)



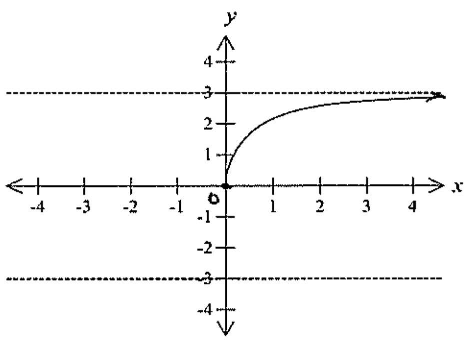
(B)



(C)



(D)



2. Let $z = 3 - i$. What is the value of $\overline{iz}$?

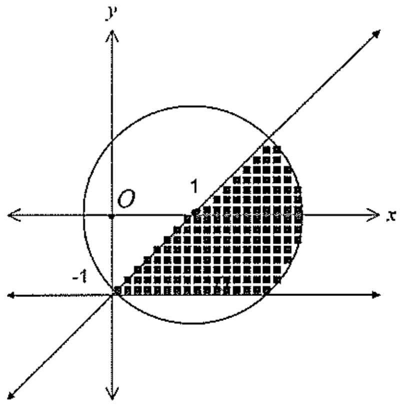
(A) $-1 - 3i$.

(B) $-1 + 3i$.

(C) $1 - 3i$.

(D) $1 + 3i$.

3. Consider the Argand diagram below.



Which inequality could define the shaded area?

(A) $|z-1| \le \sqrt{2}$ and $0 \le Arg(z-i) \le \dfrac{\pi}{4}$.

(B) $|z-1| \le \sqrt{2}$ and $0 \le Arg(z+i) \le \dfrac{\pi}{4}$.

(C) $|z-1| \le 1$ and $0 \le Arg(z-i) \le \dfrac{\pi}{4}$.

(D) $|z-1| \le 1$ and $0 \le Arg(z+i) \le \dfrac{\pi}{4}$.

4. Which of the following is an expression for $\displaystyle\int \frac{1}{\sqrt{x^2-6x+10}}\,dx$?

(A) $\ln\left(x-3-\sqrt{x^2-6x+10}\right)+c$

(B) $\ln\left(x+3-\sqrt{x^2-6x+10}\right)+c$

(C) $\ln\left(x-3+\sqrt{x^2-6x+10}\right)+c$

(D) $\ln\left(x+3+\sqrt{x^2-6x+10}\right)+c$

5. What is the solution to the inequation: $\dfrac{x(5-x)}{x-4} \ge -3$?

(A) $2 \le x < 4$ or $x \ge 6$.

(B) $1 \le x < 4$ or $x \ge 5$.

(C) $4 < x \le 6$ or $x \le 2$.

(D) $4 > x \le 5$ or $x \le 1$.

6. The points $P(a\cos\theta,\ b\sin\theta)$ and $Q(a\cos\phi,\ b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the chord PQ subtends a right angle at $(0,0)$. Which of the following is the correct expression?

(A) $\tan\theta\tan\phi=-\frac{b^2}{a^2}$.

(B) $\tan\theta\tan\phi=-\frac{a^2}{b^2}$.

(C) $\tan\theta\tan\phi=\frac{b^2}{a^2}$.

(D) $\tan\theta\tan\phi=\frac{a^2}{b^2}$.

7. What are the values of real numbers p and q such that $1-i$ is a root of the equation $z^3+pz+q=0$?

(A) $p=2$ and $q=4$.

(B) $p=2$ and $q=-4$.

(C) $p=-2$ and $q=4$.

(D) $p=-2$ and $q=-4$.

8. A particle of mass m is projected vertically upwards with an initial velocity of u ms^{-1} in a medium in which the resistance to the motion is proportional to the square of the velocity v ms^{-1} of the particle or mkv^2. Let x be the displacement in metres of the particle above the point of projection, O, so that the equation of motion is $\ddot{x}=-\left(g+kv^2\right)$ where g ms^{-2} is the acceleration due to gravity. Assume $k=10$ and the acceleration due to gravity is 10 ms^{-2}.

Which of the following gives the correct expression for the time taken?

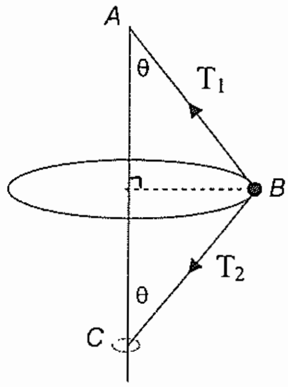
(A) $t=\frac{1}{10}\left(\tan^{-1}u-\tan^{-1}v\right)$.

(B) $t=\frac{1}{10}\left(\tan^{-1}v-\tan^{-1}u\right)$.

(C) $t=\frac{1}{10}\left(\tan^{-1}u+\tan^{-1}v\right)$.

(D) $t=\frac{1}{10}\left(\tan^{-1}v+\tan^{-1}u\right)$.

9. A body of mass m kg is attached by two light rods AB and BC. Both rods are l metres in length. Rod AB is hinged at point A and makes an angle θ with the vertical shaft. Rod BC is attached to a ring that can slide freely along the vertical shaft.



What are the tensions in the rods?

(A) $T_1 = \frac{1}{2}\left(mg\sec\theta + ml\omega^2\right)$ and $T_2 = \frac{1}{2}\left(ml\omega^2 - mg\sec\theta\right)$.

(B) $T_1 = \frac{1}{2}\left(mg\sin\theta + ml\omega^2\right)$ and $T_2 = \frac{1}{2}\left(ml\omega^2 - mg\sin\theta\right)$.

(C) $T_1 = \frac{1}{2}\left(mg\sec\theta - ml\omega^2\right)$ and $T_2 = \frac{1}{2}\left(ml\omega^2 + mg\sec\theta\right)$.

(D) $T_1 = \frac{1}{2}\left(mg\sin\theta - ml\omega^2\right)$ and $T_2 = \frac{1}{2}\left(ml\omega^2 + mg\sin\theta\right)$.

10. A skydiver falls from a plane which is flying horizontally at 2 000 m. Initially his motion is determined by the acceleration due to gravity of 10 m/s^2 and any resistance is negligible. After 5 seconds, he opens his parachute and his motion is determined by the equation: $\ddot{x} = 10 - \frac{5}{4}v$, where downwards direction is taken as positive.
Hence his terminal velocity will be 8 m/s.
Which statement best reflects the situation after the skydiver opens his parachute?

(A) He hits the ground with a vertical speed of 50 m/s.

(B) He hits the ground with a vertical speed of 8 m/s

(C) His vertical speed never exceeds 8 m/s.

(D) His vertical speed never drops below 8 m/s.

End of Section I

Section II

Total Marks is 90
Attempt Question 11 – 16.
Allow approximately 2 hours & 45 minutes for this section.

Answer all questions, starting each new question on a new sheet of paper with your **student id number** in the top right hand corner and the question number on the left hand side of your paper.

All necessary working must be shown in each and every question.

....

Question 11. [Start a New Page] **Marks**

(a) Given $z = 1+2i$ and $w = -2+i$, find:

(i) $|z|$. **1**

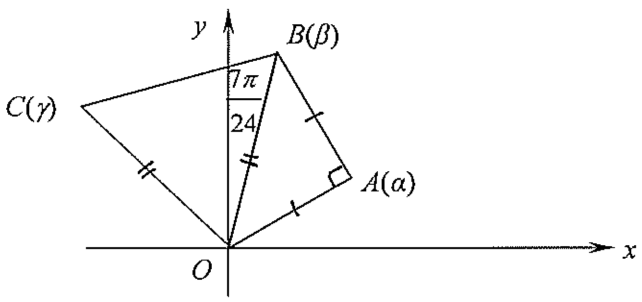
(ii) zw. **2**

(iii) $\dfrac{5}{iw}$. **2**

(b) Find the two complex numbers z that satisfy: $z\bar{z} = 37$ and $\dfrac{z}{\bar{z}} = \dfrac{35}{37} + \dfrac{12i}{37}$. **3**

(c) If $w = \left(-1 + i\sqrt{3}\right)^{2012}$, find $Arg\, w$. **2**

(d) Points A, B and C represent the complex numbers α, β and γ in the Argand diagram respectively.



Not to scale

ΔOAB is right isosceles at A, ΔCOB is isosceles with $OB = OC$ and $\angle OBC = \dfrac{7\pi}{24}$.

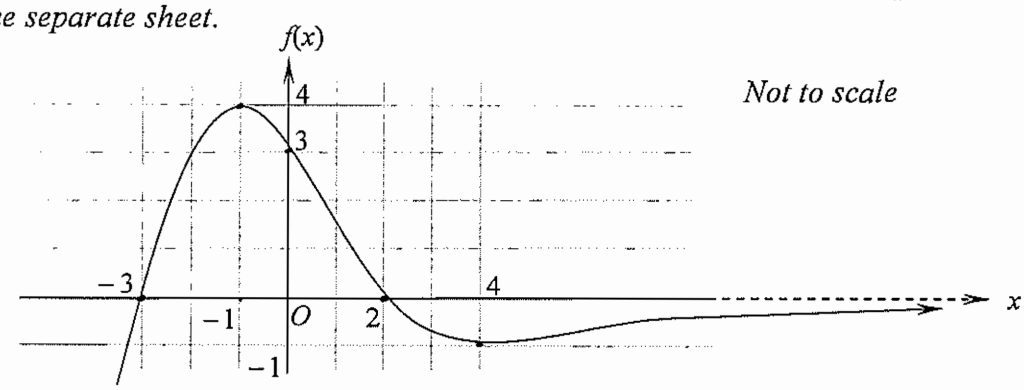
(i) Copy or trace the diagram onto your writing and find $\angle AOC$. **1**

(ii) Explain why $\gamma = \sqrt{2}\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right).\alpha$ **2**

(iii) Hence find the value of $2\alpha^2 + \gamma^2 + \alpha\gamma\sqrt{2}$. **2**

Question 12. [Start a New Page] **Marks**

(a) Given the sketch of the function $f(x)$, sketch each of the following on separate diagrams: *See separate sheet.*



(i) $y = -f(x)$. **1**

(ii) $y = f(-x)$. **2**

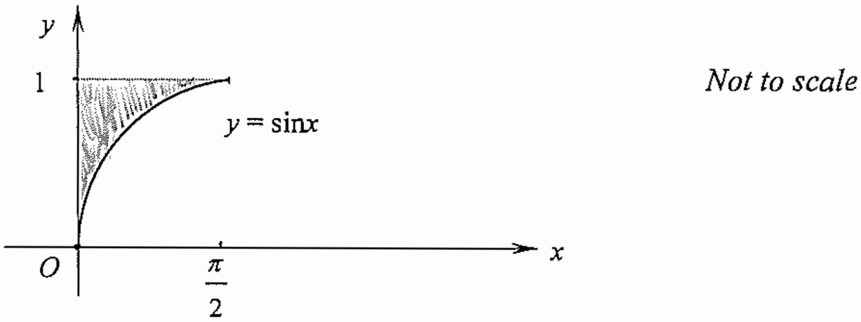
(iii) $y = f(x^2)$. **2**

(b) (i) Show that the equation of the tangent to the polynomial function $y = P(x)$ at $x = \alpha$ is $y = P'(\alpha)(x-\alpha)+P(\alpha)$. **1**

(ii) Explain why: when the polynomial $P(x)$ (of degree greater than 2) is divided by the quadratic $(x-\alpha)^2$, then $P(x) = (x-\alpha)^2 Q(x) + ax + b$, where $Q(x)$ is the quotient and a and b are real numbers. **1**

(iii) Hence show that when $P(x)$ is divided by $(x-\alpha)^2$, the remainder is equivalent to $P'(\alpha)(x-\alpha)+P(\alpha)$, the expression from the tangent in part (b) (i). **3**

(c) The area bounded by the curve $y = \sin x$, for $0 \le x \le \frac{\pi}{2}$, the lines $x = 0$ and $y = 1$ is rotated about the y-axis.



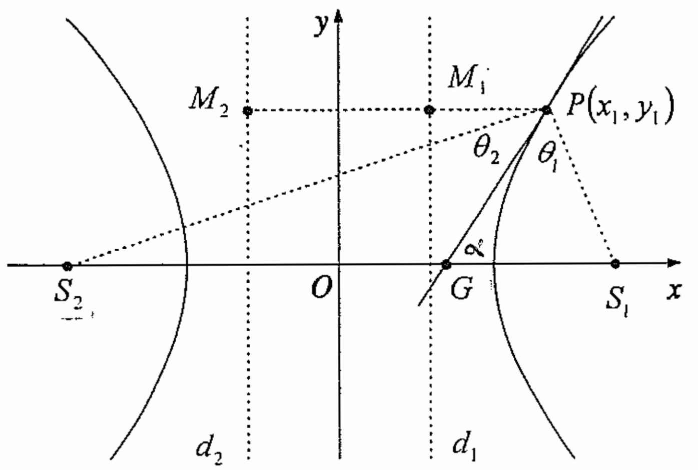
(i) By using the method of cylindrical shells, show that the volume V of the solid of revolution about the y-axis is given by: **2**

$$V = 2\pi \int_0^{\frac{\pi}{2}} x(1-\sin x)\,dx.$$

(ii) Hence calculate the volume of the solid. **3**

Question 13. [Start a New Page] **Marks**

(a)



The point $P(x_1, y_1)$ lies on the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

The two foci of the hyperbola are S_1 and S_2 and the two directrices are d_1 and d_2, as shown.

(i) Show that the length $S_1P = \frac{\sqrt{34}}{5}x_1 - 5$. **2**

(ii) Show that the equation of the tangent at P is $\frac{x_1 x}{25} - \frac{y_1 y}{9} = 1$. **2**

(iii) The tangent at P intersects the transverse axis at point G. Find the coordinates of point G. **1**

(iv) Given $\angle S_1PG = \theta_1$, $\angle GPS_2 = \theta_2$ and $\angle S_1GP = \alpha$,

(1) By using the sine rule, show that: $\sin\alpha = \frac{x_1}{5}\sin\theta_1$. **2**

(2) Hence show that: $\sin\theta_1 = \sin\theta_2$. **2**

(3) Hence deduce that GP bisects $\angle S_1PS_2$. **2**

(b) Given that $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^n x\, dx$, for $n = 1, 2, \ldots$

(i) Show that $I_1 = \frac{1}{2}\ln 2$. **1**

(ii) Show that: $I_{n-2} + I_n = \frac{1}{n-1}\left(3^{\frac{1}{2}(n-1)} - 1\right)$, for $n = 2, 3, 4, \ldots$ **2**

(iii) Find I_5. **1**

Question 14. [Start a New Page] **Marks**

(a) Given that: $\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n < \frac{\pi}{2}$, where $0 \le \theta_i < \frac{\pi}{2}$ for $i = 1, 2, 3, \ldots, n$. **3**

Prove, by mathematical induction for $n = 1, 2, 3, \ldots$, that:

$$\tan(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) \ge \tan\theta_1 + \tan\theta_2 + \tan\theta_3 + \ldots + \tan\theta_n.$$

(b) (i) Find the constants a, b and c such that: **3**

$$\frac{300x}{1000 + x^3} = \frac{a}{10 + x} + \frac{bx + c}{100 - 10x + x^2}.$$

(ii) A particle of mass m kg is projected vertically upwards in a highly resistive medium at a velocity of 5 m/s.
The particle is subjected to the force of gravity and to a resistance due to the medium of magnitude $\frac{mv^3}{100}$ newtons. Given the acceleration due to gravity is 10 m/s^2,

(1) State the equation of motion (if upwards is the positive direction) **1**

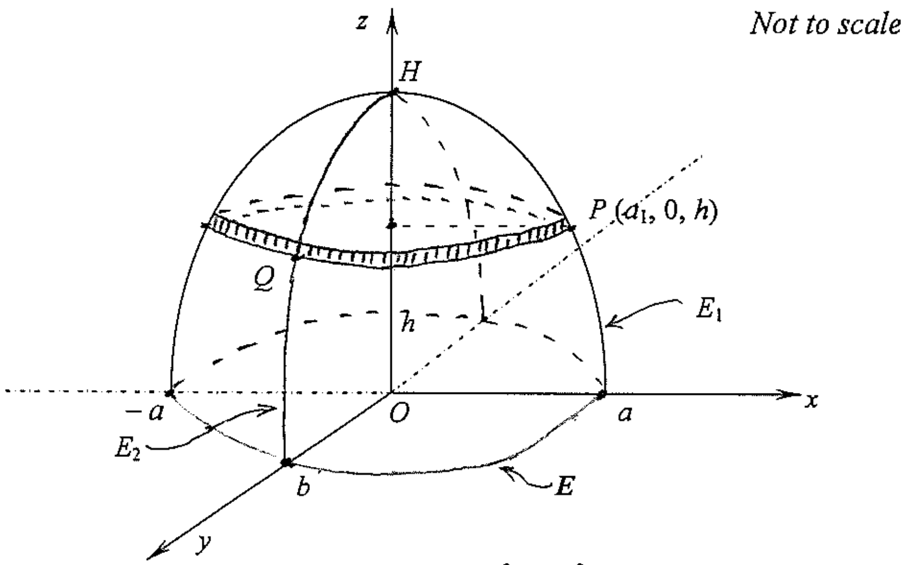
(2) Hence find the maximum height reached by the particle, (giving your answer correct to 1 decimal places). **3**

Question 14 (c) *continued over page*

(c) A right solid has an elliptical base whose equation is $\boldsymbol{E}$: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The height of the solid is H such that the sections in the x-z plane and the y-z plane are the semi-ellipses E_1 and E_2 respectively.

Every cross-section parallel to the base is elliptical in shape as shown in the diagram.



(i) Given that the equation of ellipse E_1 is $\dfrac{x^2}{a^2}+\dfrac{z^2}{H^2}=1$, **1**

State the equation for ellipse E_2.

(ii) By taking a slice parallel to the base at height of h of thickness Δh, **2**

Hence point P can be stated as $(a_1,0,h)$ and Q as $(0,b_1,h)$ for the elliptical slice, as shown in the diagram.

By assuming that the area of ellipse $\boldsymbol{E}$ is πab square units,

Show that the cross-sectional area A of this slice is given by $A=\pi ab\left(1-\dfrac{h^2}{H^2}\right)$.

(iii) Hence find the volume of the solid. **2**

Question 15. [Start a New Page] **Marks**

(a) Let α, β and γ be the distinct roots of the cubic equation $x^3 + ax^2 + bx - 54 = 0$, where a and b are real numbers. Suppose that $\alpha^2 + \beta^2 = 0$ and $\alpha^2 + \gamma^2 = 0$,

(i) Explain why $\beta + \gamma = 0$. **2**

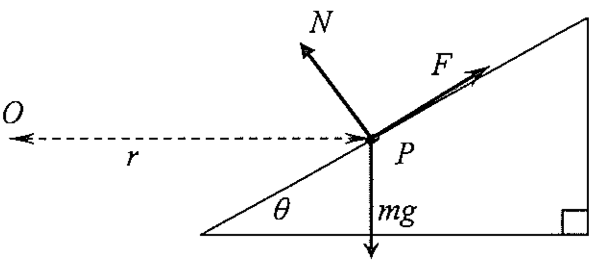
(ii) Hence explain why α is real. **2**

(iii) Hence, or otherwise explain why β and γ are complex and purely imaginary. **2**

(iv) Find a and b. **2**

(b) On a racetrack for small cars of mass m kg, a circular bend of radius r m is banked at an angle of θ to the horizontal.
The maximum frictional force is F Newtons (up or down the bank) and the acceleration due to gravity is 10 m/s^2 ie $g = 10$. The normal reaction to the surface is N Newtons.
Let point P represent the small car on the banked track as shown in the diagram.

Not to scale



(i) From the diagram, the vertical resolution for motion downwards at P is: **1**

$$N\cos\theta + F\sin\theta = mg.$$

Find the horizontal resolution when the car is travelling at speed v m/s at P.

(ii) Hence, if $r = 80$, $\theta = 45^0$ and the maximum frictional force is at most $\frac{1}{9}$ of the normal reaction force N. Find the minimum speed that the car can safely negotiate the bend without slipping down the incline. **2**

(iii) For the upwards motion of the car, find the maximum speed that the car can safely negotiate the bend. **3**

(iv) Hence or otherwise, determine the designed speed (no slipping) for this angle. **1**

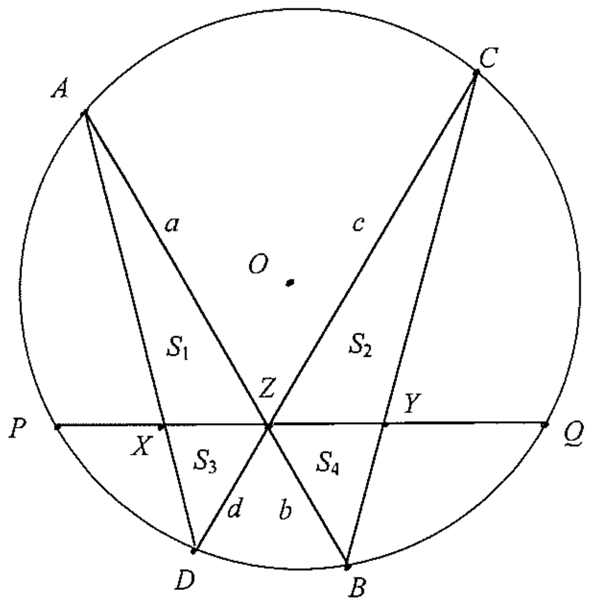
Question 16. [Start a New Page] **Marks**

(a) Show that $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$, using the substitution $x = \frac{1-u}{1+u}$. **4**

(b) The roots of $x^4 + 3x - 1 = 0$ are α, β, γ and δ. Find $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. **2**

(c)

Not to scale



Given Z is any point on the chord PQ of the given circle. Chord AB and CD pass through Z. Let X and Y be the points of intersection of the chords AD and CB with PQ respectively, as shown in the diagram.

Given $ZA = a$; $ZB = b$; $ZC = c$ and $ZD = d$. **Let** $ZP = p$; $ZQ = q$; $ZX = x$ and $ZY = y$.

Given S_1 denotes the *area* of ΔZAX; S_2 for ΔZCY; S_3 for ΔZDX and S_4 for ΔZBY.

(i) By detaching page **12** and stapling it to your Question 16,
Show that: $\frac{S_1}{S_2} = \frac{a.AX}{c.CY}$ and $\frac{S_1}{S_4} = \frac{a.x}{b.y}$. **3**

(ii) Hence deduce that: $\frac{S_1 S_3}{S_2 S_4} = \frac{a.d \times AX.XD}{b.c \times CY.YB} = \frac{a.d.x^2}{b.c.y^2}$. **2**

(iii) Hence explain why: $\frac{x^2}{y^2} = \frac{(p-x)(x+q)}{(p+y)(q-y)}$. **1**

(iv) Hence show that: $\frac{1}{x} - \frac{1}{y} = \frac{1}{p} - \frac{1}{q}$. **2**

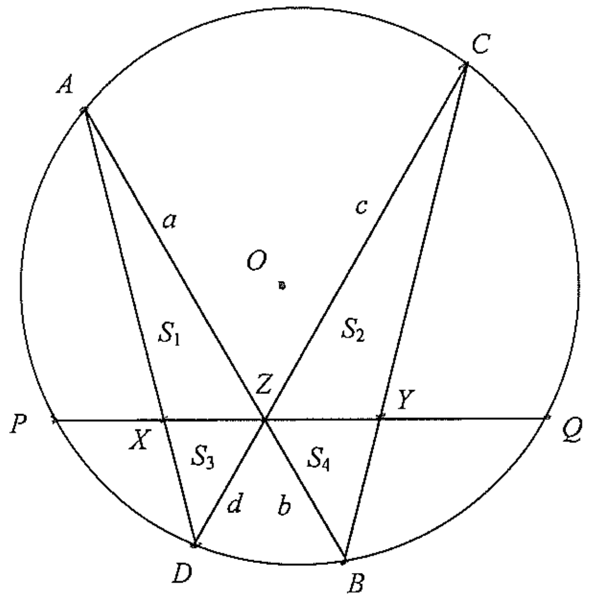
(v) If Z is the midpoint of PQ, what is the relationship between x and y. **1**

THE END

👍 ☺ 😐 ☹

Student id No.				

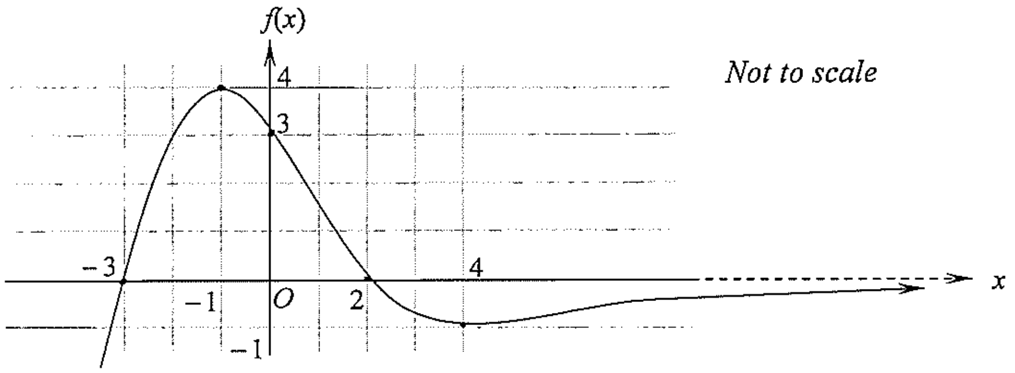
Question 16. (c) Attach this diagram to your **Q16** (c) solutions.



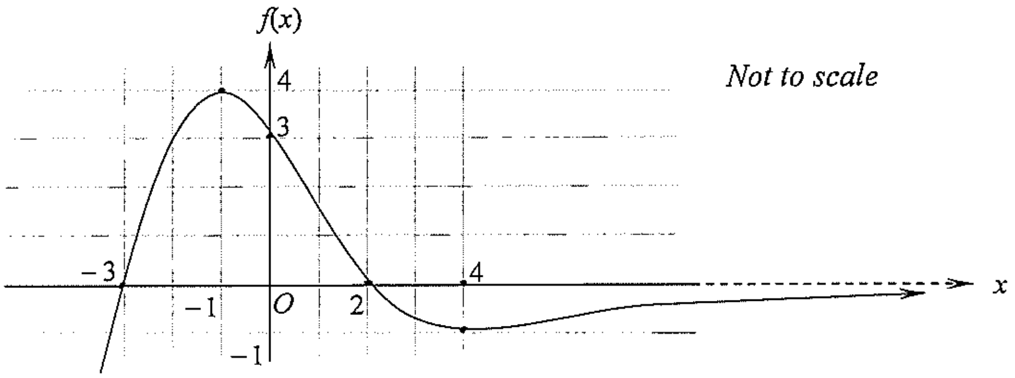
M Ext 2
Question 12 (a)

Student id No.				

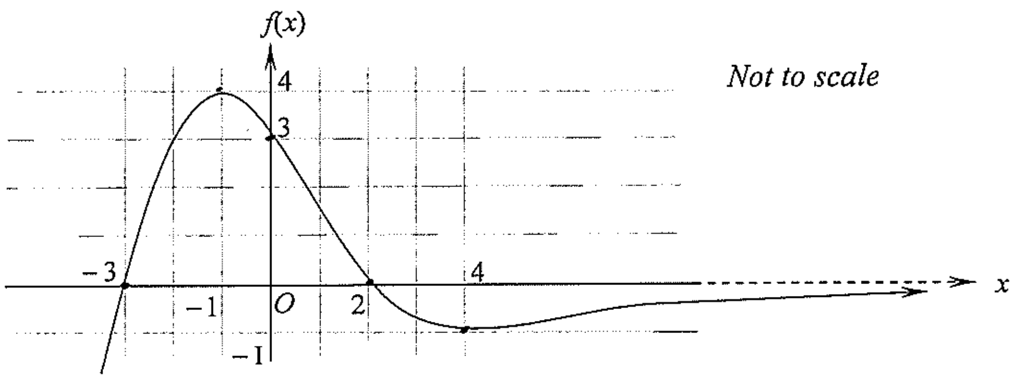
(i)



(ii)



(iii)



Section I

10 Marks
Attempt Question 1 – 10.
Allow approximately 15 minutes for this section.

Use the multiple choice answer sheet below to record your answers to Question 1 – 10.

Select the alternative: A, B, C or D that best answers the question.

Colour in the response oval completely.

Sample:

$2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, draw a cross through the incorrect answer and colour in the new answer

ie A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word "*correct*" and draw an arrow as follows:

correct

A ● B ⊗ C ○ D ○

Trial HSC Examination
Mathematics Extension 2, 2012

Multiple Choice Answer Sheet

SOLUTIONS

Student id number: ANSWERS.

Completely colour in the response oval representing the most correct answer.

1	A	○	B	○	C	●	D	○
2	A	○	B	○	C	●	D	○
3	A	○	B	●	C	○	D	○
4	A	○	B	○	C	●	D	○
5	A	○	B	○	C	●	D	○
6	A	○	B	●	C	○	D	○
7	A	○	B	○	C	●	D	○
8	A	●	B	○	C	○	D	○
9	A	●	B	○	C	○	D	○
10	A	○	B	○	C	○	D	●

Mark: /10

JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 11

Suggested Solutions	Marks	Marker's Comments

Q11. (a) (i) $z = 1+2i$, $w = -2+i$

$|z| = \sqrt{1^2+2^2}$

$\therefore |z| = \sqrt{5}$

$\boxed{1}$

(ii) $zw = (1+2i)(-2+i)$

$= -2+i-4i+2i^2 = -2+i-4i-2$

$\therefore zw = -4-3i$

$\boxed{2}$

(iii) $\dfrac{5}{iw} = \dfrac{5}{i(-2+i)} = \dfrac{5}{-1-2i} \times \dfrac{-1+2i}{-1+2i}$

$= \dfrac{5(-1+2i)}{(-1)^2+4} = \dfrac{5(-1+2i)}{5}$

$\therefore \dfrac{5}{iw} = -1+2i$

$\boxed{2}$

Comment: $\dfrac{5}{iw} \times \dfrac{\overline{iw}}{\overline{iw}} = \dfrac{-5i\bar{w}}{|w|^2}$

$= -\dfrac{5i(-2-i)}{5} =$

(b) $z\bar{z} = 37$ $\quad \dfrac{z}{\bar{z}} = \dfrac{35}{37} + \dfrac{12i}{37}$

Now $\dfrac{z}{\bar{z}} \times \dfrac{z}{z} = \dfrac{z^2}{z\bar{z}} = \dfrac{z^2}{37}$

$\therefore \dfrac{z^2}{37} = \dfrac{35}{37} + \dfrac{12i}{37} \Rightarrow z^2 = 35+12i$

Let $z = x+iy$ $\quad x, y \in \mathbb{R}$

$z^2 = x^2 - y^2 + 2xyi$

$z\bar{z} = x^2 + y^2$

$\therefore x^2 - y^2 = 35 \quad — (1)$

$2xy = 12$

$xy = 6 \quad — (2)$

$x^2 + y^2 = 37 \quad — (3)$

(1) and (3) $\quad 2x^2 = 72$

$x^2 = 36$

$x = \pm 6$

sub in (2) $\quad y = \dfrac{6}{x} = 1$ or -1

$\therefore z = 6+i$ or $-6-i$ $\qquad \pm(6+i)$

$\boxed{3}$

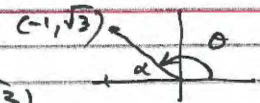
Comment: $z = x+iy$

$\bar{z} = x - iy$

$z\bar{z} = x^2 + y^2$

$z^2 = x^2 - y^2 + 2xyi$

(c) $w = (-1+i\sqrt{3})^{2012}$



$\tan\alpha = \dfrac{\sqrt{3}}{1} \Rightarrow \alpha = \dfrac{\pi}{3}$

$\therefore \theta = \dfrac{2\pi}{3}$

$\operatorname{Arg} w = 2012 \operatorname{Arg}(-1+i\sqrt{3})$

$= 2012 \times \dfrac{2\pi}{3} = \dfrac{4024\pi}{3} \quad (\div 2\pi)$

$= -\dfrac{2\pi}{3}$

$\boxed{2}$

Comment: $\dfrac{2012}{3}$

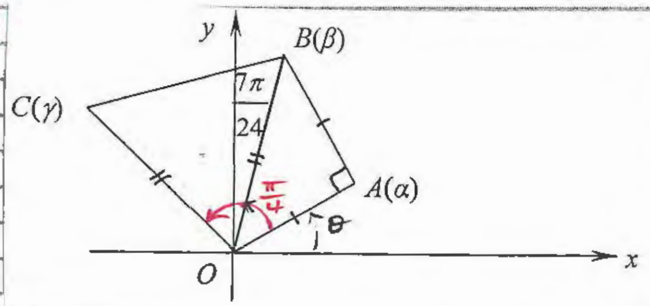
\\TITAN\StaffHome$\woh08\JRAH M Fac Admin\Assessment info\Suggested Mk solns template_V4_half Ls.doc

JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 11

Suggested Solutions	Marks	Marker's Comments

(d)



(i)

$\angle AOB = \frac{\pi}{4}$ right isosceles ΔOAB

$\angle OCB = \frac{7\pi}{24}$ (equal angles opposite equal sides $OB = OC$)

$2 \times \frac{7\pi}{24} + \angle COB = \pi$ (Angle sum of ΔCOB is π)

$\angle COB = \pi - \frac{7\pi}{12} = \frac{5\pi}{12}$

$\therefore \angle AOC = \frac{\pi}{4} + \frac{5\pi}{12} = \frac{8\pi}{12} = \frac{2\pi}{3}$ — 1

(ii)

$\vec{OC} = \gamma$, $\operatorname{Arg}\gamma = \theta + \angle AOC = \theta + \frac{2\pi}{3}$, $\operatorname{Arg}\alpha \equiv \theta$ — Enlarge by and Rotate by

$OC = OB = |\beta| = \sqrt{2}.OA$ (right isosceles Δ, ... Pyth. Thm)

but $OC = |\gamma| = |\beta| = \sqrt{2}\,OA$

$\therefore \gamma = \beta \operatorname{cis}\frac{2\pi}{3} = \sqrt{2}.\alpha.\operatorname{cis}\frac{2\pi}{3} = \sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\alpha.$ — 2

(iii)

$LHS = 2\alpha^2 + \gamma^2 + \alpha\gamma\sqrt{2} = 2\alpha^2 + \left[\sqrt{2}\operatorname{cis}\frac{2\pi}{3}\alpha\right]^2 + \alpha.\sqrt{2}\operatorname{cis}\frac{2\pi}{3}.\alpha.\sqrt{2}$

$= 2\alpha^2 + 2\alpha^2\operatorname{cis}\frac{4\pi}{3} + 2\alpha^2\operatorname{cis}\frac{2\pi}{3}$ (de Moivres thm)

$= 2\alpha^2\left[1 + \operatorname{cis}\frac{4\pi}{3} + \operatorname{cis}\frac{2\pi}{3}\right]$

$= 2\alpha^2\left[1 + -\frac{1}{2} - \frac{\sqrt{3}}{2}i + -\frac{1}{2} + \frac{i\sqrt{3}}{2}\right]$

$= 0$ — 2

\\TITAN\StaffHome$\woh08\JRAH M Fac Admin\Assessment info\Suggested Mk solns template_V4_half Ls.doc

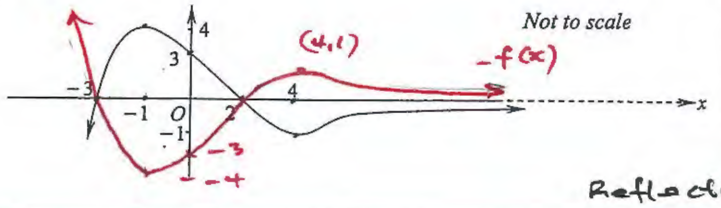
JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 12

Suggested Solutions	Marks	Marker's Comments

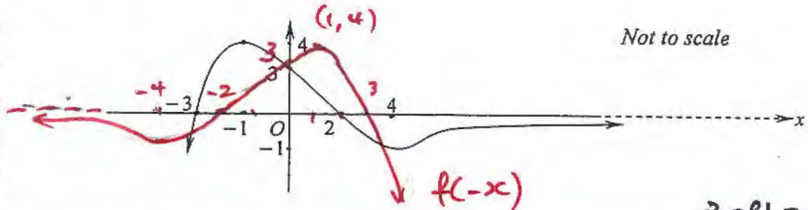
(a) **Question 12 (a)**

(i)



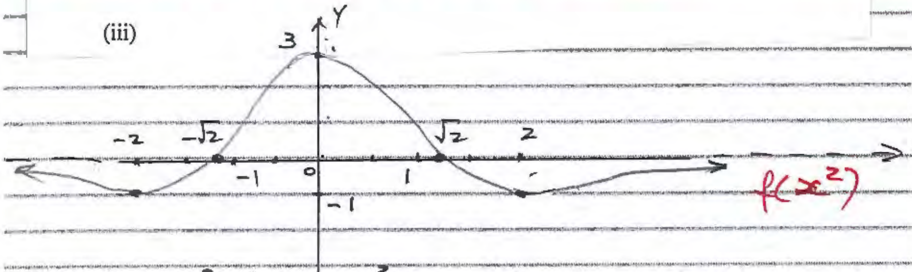
Reflection in x-axis [1]

(ii)



Reflection in y-axis. [2]

(iii)



New x-int: $x^2 = -3 \,;\, -1 \,;\, 2$

$\therefore \quad x = \cancel{\ } \,;\, \cancel{\ } \,;\, \pm\sqrt{2}$ [2]

$x = 0 \quad x^2 = 0 \quad f(x^2) = f(0) = 3 \qquad (0, 3)$

$x = \pm 2 \quad x^2 = 4 \quad f(x^2) = f(4) = -1. \qquad (\pm 2, -1)$

(b) (i) $y = P(x) \quad \therefore \dfrac{dy}{dx} = P'(x)$

Gradient of Tangent at $(\alpha, P(\alpha))$ $m_T = P'(\alpha)$

Eqn of Tangent : $y - P(\alpha) = P'(\alpha)(x - \alpha)$

ie $y = P'(\alpha)(x - \alpha) + P(\alpha)$ QED [1]

(ii) $P(x) = (x - \alpha)^2 Q(x) + R(x)$.

Since the divisor $(x-\alpha)^2$ is of degree 2

$\therefore$ deg of remainder $R(x)$ is at most 1. [1]

$\therefore$ $R(x)$ is linear

ie $R(x) = ax + b$

MATHEMATICS Extension 2: Question 12

Suggested Solutions	Marks	Marker's Comments

(b)(iii) $P(x) = (x-\alpha)^2 Q(x) + ax + b$

$\therefore \quad P(\alpha) = 0 + a\alpha + b$

$P(\alpha) = a\alpha + b \quad \text{— (1)}$

$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x) + a$

$\therefore \quad P'(\alpha) = 0 + 0 + a$

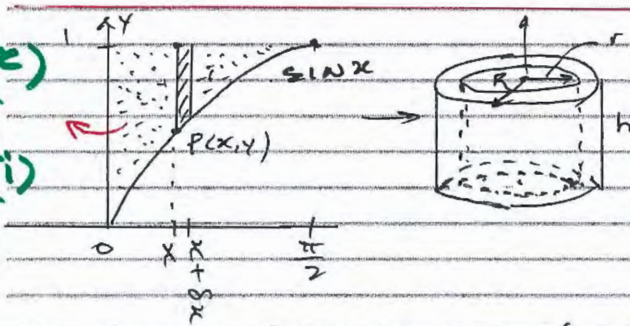
$\therefore \quad a = P'(\alpha) \quad \text{— (2)}$

so $R(x) = ax + b$
$= P'(\alpha)x + P(\alpha) - a\alpha$
$= P'(\alpha)x - P'(\alpha)\alpha + P(\alpha)$
$= P'(\alpha)(x-\alpha) + P(\alpha)$

$\therefore$ Remainder is the same as the tangent at $x=\alpha$ when divide by $(x-\alpha)^2$.

[3]

(c)(i)



$r = x$
$R = x + \delta x$

$A \doteqdot \pi[R^2 - r^2] = \pi[(R+r)(R-r)]$

$A \doteqdot \pi[(2x+\delta x)(\delta x)] = \pi[2x\delta x + (\delta x)^2]$

$A \doteqdot 2\pi x\,\delta x$, neglect $(\delta x)^2 \doteqdot 0$

$\therefore$ Volume of shell $\delta V \doteqdot 2\pi x\,\delta x(1-y) = 2\pi x(1-y)\,\delta x$

Volume of solid $V \doteqdot \sum_{1}^{n} 2\pi x(1-\sin x)\,\delta x$

$= \lim_{\substack{\delta x \to 0 \\ n \to \infty}} \sum_{0}^{\pi/2} 2\pi x(1-\sin x)\,\delta x$

$= 2\pi \int_0^{\pi/2} x(1-\sin x)\,dx.$

[2]

Marker's Comments:
"$r = x - \frac{1}{2}\delta x$"
"$R = x + \frac{1}{2}\delta x$"
$A \doteqdot \pi \times 2x \times \delta x$

"Rectangular Prism"
$A \doteqdot 2\pi x \times (1-y)$
$\delta V \doteqdot 2\pi x(1-y)\,\delta x$

(ii) $V = 2\pi \int_0^{\pi/2} x(1-\sin x)\,dx \qquad u = x,\ du = dx \qquad dv = (1-\sin x)\,dx,\ V = x + \cos x$

$= 2\pi\left\{[x(x+\cos x)]_0^{\pi/2} - \int_0^{\pi/2}(x+\cos x)\,dx\right\}$

$= 2\pi\left\{\frac{\pi}{2}\cdot\frac{\pi}{2} - 0 - \left[\frac{x^2}{2} + \sin x\right]_0^{\pi/2}\right\}$

$= 2\pi\left\{\frac{\pi^2}{4} - \left(\frac{\pi^2}{8} + 1\right) - 0\right\}$

$= 2\pi\left\{\frac{\pi^2}{4} - \frac{\pi^2}{8} - 1\right\} = 2\pi\left\{\frac{\pi^2}{8} - 1\right\}$

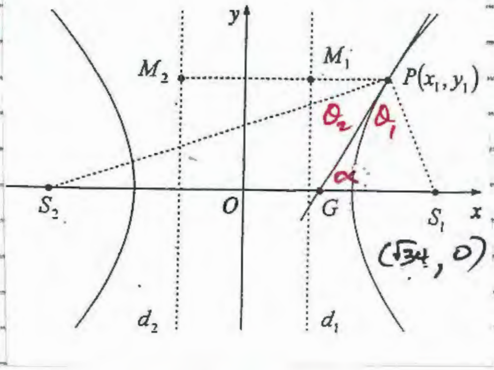
Volume is $\left(\frac{\pi^3}{4} - 2\pi\right) u^3$

[3]

METHOD I
$2\pi\left[\frac{x^2}{2} + x\cos x - \sin x\right]_0^{\pi/2}$

JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 13

Suggested Solutions	Marks	Marker's Comments
(a) (i)  $\frac{x^2}{25} - \frac{y^2}{9} = 1$ $a=5$, $b=3$		Distance Formula $S_1P = \sqrt{(\sqrt{34}-x_1)^2 + (0-y_1)^2}$
$e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{9}{25} = \frac{34}{25}$		
$e = \frac{\sqrt{34}}{5}$; $e>1$		
$S_1(ae, 0) = (\sqrt{34}, 0)$		
d_1: $x = \frac{a}{e} = \frac{25}{\sqrt{34}}$		
$\therefore S_1P = e\,PM_1$ (focus-directrix defn.)		
$= \frac{\sqrt{34}}{5}\left(x_1 - \frac{25}{\sqrt{34}}\right)$		
$S_1P = \frac{\sqrt{34}}{5}x_1 - 5$ qed.	2	
(ii) $\frac{x^2}{25} - \frac{y^2}{9} = 1$		
$\frac{d}{dx}\left[\frac{x^2}{25} - \frac{y^2}{9}\right] = \frac{d}{dx}1$		
$\frac{2x}{25} - \frac{2yy'}{9} = 0$		
$\frac{dy}{dx} = \frac{9x}{25y}$		
Gradient of tangent at P : $m_T = \frac{9x_1}{25y_1}$		
Equ. of Tangent $y - y_1 = \frac{9x_1}{25y_1}(x - x_1)$		
$\times \frac{y_1}{9}$ $\quad \frac{y_1y}{9} - \frac{y_1^2}{9} = \frac{x_1x}{25} - \frac{x_1^2}{25}$		
so $\frac{x_1x}{25} - \frac{y_1y}{9} = \frac{x_1^2}{25} - \frac{y_1^2}{9}$	2	
ie $\frac{x_1x}{25} - \frac{y_1y}{9} = 1$ qed as $\frac{x_1^2}{25} - \frac{y_1^2}{9} = 1$		as $P(x_1, y_1)$ is on Hyp.
(iii) For G $y=0$ $\therefore \frac{x_1x}{25} = 1$		
ie $x = \frac{25}{x_1}$		
ie $G = \left(\frac{25}{x_1}, 0\right)$	1	
(iv) For ΔS_1PG : $\frac{S_1P}{\sin\alpha} = \frac{S_1G}{\sin\theta_1}$		$S_1G = \sqrt{34} - \frac{25}{x_1}$
(1) $\sin\alpha = \frac{S_1P\sin\theta_1}{S_1G} = \frac{\left(\frac{\sqrt{34}}{5}x_1 - 5\right)\sin\theta_1}{\sqrt{34} - \frac{25}{x_1}}$		
$= \frac{(\sqrt{34}\,x_1 - 25)\,x_1\sin\theta_1}{(\sqrt{34}\,x_1 - 25)\,5}$;		$x_1 \neq \frac{25}{\sqrt{34}}$.
$\therefore \sin\alpha = \frac{x_1}{5}\sin\theta_1$ qed.	2	

JRAHS M. EXT 2 TRIAL, 2012

SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 13

Suggested Solutions	Marks	Marker's Comments

(iv) (2) In $\triangle S_2PG$, $\angle PGS_2 = \pi - \alpha$

$$\frac{S_2P}{\sin(\pi-\alpha)} = \frac{S_2G}{\sin\theta_2}$$

$$\frac{S_2P}{\sin\alpha} = \frac{S_2G}{\sin\theta_2}$$

$$\sin\alpha = \frac{S_2P\sin\theta_2}{S_2G}$$

$S_2P = ePM_2 = \frac{\sqrt{34}}{5}x_1 + 5 = \frac{\sqrt{34}x_1 + 25}{5}$

$S_2G = \frac{25}{x_1} + \sqrt{34} = \frac{\sqrt{34}x_1 + 25}{x_1}$

$$= \frac{x_1}{5}\sin\theta_2$$

so $\sin\alpha = \frac{x_1}{5}\sin\theta_1 = \frac{x_1}{5}\sin\theta_2$

$\therefore \sin\theta_1 = \sin\theta_2$

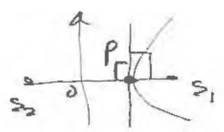
$\boxed{2}$

(3) $\sin\theta_1 = \sin\theta_2$

$\therefore \theta_1 = \theta_2$ or $\theta_1 = \pi - \theta_2$

Now $\theta_1 = \pi - \theta_2$ only when $\theta_1 = \theta_2 = \frac{\pi}{2}$

ie. P is at Vertex A



$\therefore \theta_1 \equiv \theta_2$

$\therefore$ GP bisects $\angle S_1PS_2$

$\boxed{2}$

(b) $I_n = \int_{\pi/6}^{\pi/4} \cot^n x\, dx \quad n = 1, 2, 3, \ldots$

$\sin x > 0$ $[\frac{\pi}{6}, \frac{\pi}{4}]$

(i) $I_1 = \int_{\pi/6}^{\pi/4} \frac{\cos x}{\sin x}\, dx = \ln(\sin x)\Big]_{\pi/6}^{\pi/4}$

$$= \ln\frac{1}{\sqrt{2}} - \ln\frac{1}{2} = \ln\frac{2}{\sqrt{2}} = \ln\sqrt{2} = \ln 2^{\frac{1}{2}}$$

$I_1 = \frac{1}{2}\ln 2$ qed

$\boxed{1}$

(ii) $I_{n-2} + I_n = \int \cot^{n-2} x + \cot^n x\, dx$

$$= \int \cot^{n-2} x\,(1 + \cot^2 x)\, dx$$

$$= \int \cot^{n-2} x \cdot \operatorname{cosec}^2 x\, dx$$

$$= -\frac{1}{n-1}\cot^{n-1} x\Big]_{\pi/6}^{\pi/4}$$

$$= -\frac{1}{n-1}\left[1 - (\sqrt{3})^{n-1}\right] = \frac{1}{n-1}\left[3^{\frac{1}{2}(n-1)} - 1\right]$$

$\boxed{2}$

(iii) I_5: $n = 5$ $I_3 + I_5 = \frac{1}{4}[3^2 - 1] = 2$

$n = 3$ $I_1 + I_3 = \frac{1}{2}[3 - 1] = 1$

$\therefore I_5 = 2 - I_3 = 2 - (1 - I_1) = 1 + I_1$

$= 1 + \frac{1}{2}\ln 2.$

$\boxed{1}$

JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 14

Suggested Solutions	Marks	Marker's Comments

(a) Let $S_n = \theta_1 + \theta_2 + \dots + \theta_n < \frac{\pi}{2}$

Let $P(n)$: $\tan(S_n) \geq \tan\theta_1 + \tan\theta_2 + \dots + \tan\theta_n$

- For $P(1)$:

$LHS = \tan\theta_1$ $\quad$ $RHS = \tan\theta_1$

$\therefore P(1)$ is true

- Assume $P(n)$, is true up to some integer $k \geq 1$

ie $S_k = \theta_1 + \theta_2 + \dots + \theta_k < \frac{\pi}{2}$ and

$\tan(S_k) \geq \tan\theta_1 + \tan\theta_2 + \dots + \tan\theta_k$ ——— (1)

RTP $P(k+1)$ is true

ie $\tan(S_{k+1}) \geq \tan\theta_1 + \tan\theta_2 + \dots + \tan\theta_k + \tan\theta_{k+1}$

- Proof $P(k+1)$

Now $\tan S_{k+1} = \tan(S_k + \theta_{k+1})$

$= \dfrac{\tan S_k + \tan\theta_{k+1}}{1 - \tan S_k \cdot \tan\theta_{k+1}}$ $\quad$ * using assumption (1)

$\geq \dfrac{\tan\theta_1 + \tan\theta_2 + \dots + \tan\theta_k + \tan\theta_{k+1}}{1 - \tan S_k \cdot \tan\theta_{k+1}}$

Now $\theta_1 + \theta_2 + \dots + \theta_k + \theta_{k+1} < \frac{\pi}{2}$

$\therefore S_k < \frac{\pi}{2} - \theta_{k+1}$ $\quad$ $0 \leq \theta_i < \pi/2$

$\therefore \tan S_k < \tan\left(\frac{\pi}{2} - \theta_{k+1}\right) = \dfrac{1}{\tan\theta_{k+1}}$

$0 \leq \tan S_k \tan\theta_{k+1} < 1.$

$\therefore 0 < 1 - \tan S_k \tan\theta_{k+1} < 1$

$\therefore \dfrac{1}{1 - \tan S_k \tan\theta_{k+1}} > 1$

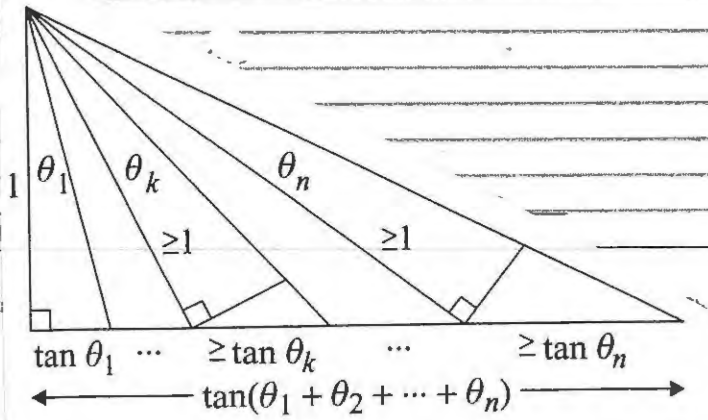
3

Hence $\tan S_{k+1} \geq \tan\theta_1 + \tan\theta_2 + \dots + \tan\theta_{k+1}$

$\therefore P(k+1)$ is true

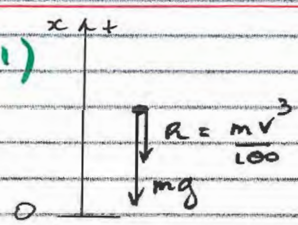
$\therefore$ By the PMI $P(n)$ is true for $n = 1, 2, 3, \dots$

NOTE:



JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

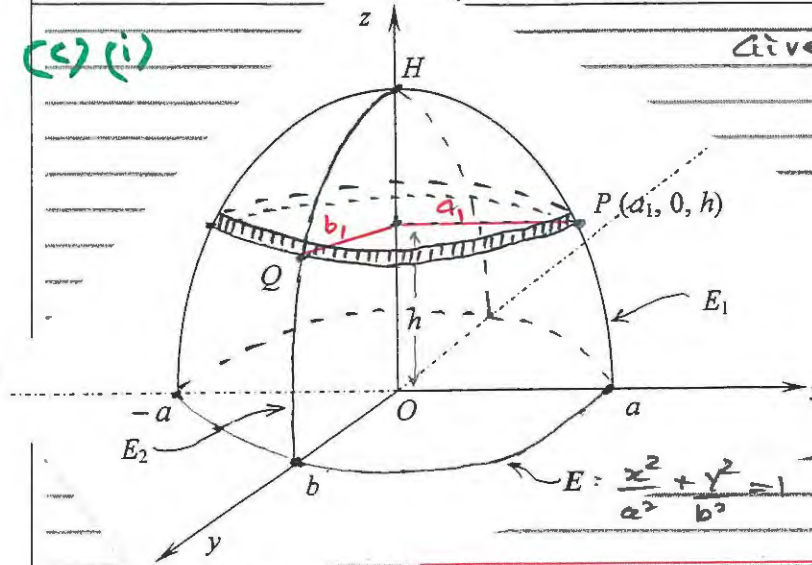
MATHEMATICS Extension 2: Question 14

Suggested Solutions	Marks	Marker's Comments
(b)(i) $\dfrac{300x}{1000+x^3} = \dfrac{a}{10+x} + \dfrac{bx+c}{100-10x+x^2}$		
$\therefore\ 300x \equiv a(100-10x+x^2) + (bx+c)(10+x)$		
$\bullet\ x=-10:\ -3000 = 300a$		
$\therefore\ \underline{a=-10}$		
$\bullet\ x=0:\ 0 = 100a + 10c$		
$10c = -100a = 1000$		
$\underline{c=100}$		
$\bullet\ x=10:\ 3000 = 100a + (10b+c)\times 20$		
$= -1000 + 20(10b+100)$	3	
$40000 = 200b + 2000$		
$\therefore\ \underline{b=10}$		
$\Rightarrow \begin{cases} a=-10 \\ b=10 \\ c=100 \end{cases}$		
(ii)(1)  Data $t=0,\ x=0,\ v=5,\ g=10,\ \ddot{x}=?$		
Equation of motion		
$m\ddot{x} = -mg - \dfrac{mv^3}{100}$	1	
$\ddot{x} = -10 - \dfrac{v^3}{100} = -\dfrac{1}{100}(1000+v^3)$		
(2) $\ddot{x} = v\dfrac{dv}{dx} = -\dfrac{1}{100}(1000+v^3)$		
Max height when $x=H$ $v=0$		
$\displaystyle\int_5^0 \frac{100v\,dv}{1000+v^3} = -\int_0^H dx$		
so $\displaystyle\frac{1}{3}\int_5^0 \frac{300v\,dv}{1000+v^3} = \frac{1}{3}\int_5^0 \frac{-10}{10+v} + \frac{10v+100}{100-10v+v^2}\,dv = -[x]_0^H$		
ie $\displaystyle\frac{10}{3}\int_5^0 \frac{-1}{10+v} + \frac{v+10}{100-10v+v^2}\,dv = -H$		
$\displaystyle\frac{10}{3}\int_5^0 \frac{-1}{10+v} + \frac{\frac{1}{2}(2v-10)}{v^2-10v+100} + \frac{15}{75+(v-5)^2}\,dv = -H$		$\frac{15}{5\sqrt{3}} = \sqrt{3}$
$\dfrac{10}{3}\left[-\ln(10+v) + \dfrac{1}{2}\ln(v^2-10v+100) + \dfrac{15}{5\sqrt{3}}\tan^{-1}\dfrac{(v-5)}{5\sqrt{3}}\right]_5^0 = -H$		
$-H = \dfrac{10}{3}\left[\left(-\ln 10 + \dfrac{1}{2}\ln 100 + \sqrt{3}\tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)\right) - \left(-\ln 15 + \dfrac{1}{2}\ln 75 + 0\right)\right]$	3	
$-H = \dfrac{10}{3}\left[-\sqrt{3}\cdot\dfrac{\pi}{6} + \ln 15 - \dfrac{1}{2}\ln 75\right]$		$\therefore$ Max height is $\underline{1{\cdot}2}$ m (1dp)
$H = \dfrac{10}{3}\left[\dfrac{\pi\sqrt{3}}{6} + \dfrac{1}{2}\ln\dfrac{5}{3}\right] = 1{\cdot}19197\ldots$	8.	

JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 14

Suggested Solutions	Marks	Marker's Comments
(c) (i) Given $E_1: \frac{x^2}{a^2}+\frac{z^2}{H^2}=1$ so $E_2: \frac{y^2}{b^2}+\frac{z^2}{H^2}=1$		1
(ii) Since P lies on E_1 at $z=h$ $\frac{a_1^2}{a^2}+\frac{h^2}{H^2}=1$ ie $a_1^2=a^2\left(1-\frac{h^2}{H^2}\right)$ And Q lies on E_2 at $z=h$ $\frac{b_1^2}{b^2}+\frac{h^2}{H^2}=1$ ie $b_1^2=b^2\left(1-\frac{h^2}{H^2}\right)$ $\therefore$ Area of slice at $z=h$: $A \doteqdot \pi a_1 b_1$ ie $A \doteqdot \pi a\sqrt{1-\frac{h^2}{H^2}} \times b\sqrt{1-\frac{h^2}{H^2}}$ $A=\pi ab\left(1-\frac{h^2}{H^2}\right)$ qed		2
(iii) Volume of slice $\delta V \doteqdot \pi ab\left(1-\frac{h^2}{H^2}\right)\times \delta h$ Volume of solid $V \doteqdot \sum_1^n \pi ab\left(1-\frac{h^2}{H^2}\right)\delta h$ $=\lim_{\substack{\delta h\to 0\\ n\to\infty}} \sum_0^H \pi ab\left(1-\frac{h^2}{H^2}\right)\delta h$ $V=\pi ab\int_0^H 1-\frac{h^2}{H^2}\,dh$ $=\pi ab\left[h-\frac{h^3}{3H^2}\right]_0^H$ $V=\pi ab\left[H-\frac{H^3}{3H^2}\right]=\frac{2}{3}\pi abH$		Let $\delta h \equiv \Delta h$ 2
$\therefore$ Volume is $\frac{2}{3}abH\pi\ u^3$		



JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 15

Suggested Solutions	Marks	Marker's Comments
(a) $x^3 + ax^2 + bx - 54 = 0$		
(i) As $\alpha^2 + \beta^2 = 0$ — (1) $\alpha^2 + \gamma^2 = 0$ — (2) $\therefore \beta^2 = \gamma^2 \Rightarrow \beta^2 - \gamma^2 = 0$ $\beta = \pm\gamma$ but α, β and γ are distinct $\therefore \beta \neq \gamma$ $\therefore \beta = -\gamma \Rightarrow \beta + \gamma = 0.$		$\boxed{2}$
(ii) Now $s_1 = \Sigma\alpha = \alpha + \beta + \gamma = -a$ ie $\alpha + 0 = -a$ $\alpha = -a$ As a is real (data) $\therefore \alpha$ is real		$a \in \mathbb{R}$ $\alpha = -a$ $\Rightarrow \alpha \in \mathbb{R}$ $\boxed{2}$
(iii) Now $\alpha^2 + \beta^2 = 0$ $\beta^2 = -\alpha^2 = i^2\alpha^2$ $\beta = \pm i\alpha$ As $\alpha \in \mathbb{R}$ $\therefore \beta$ is purely imaginary As $\gamma = -\beta = \mp i\alpha$. So again γ is purely imaginary		$\boxed{2}$
(iv) Now $s_3 = \Sigma\alpha\beta\gamma = \alpha\beta\gamma = -(-54) = 54$ but $\gamma = -\beta$ $\therefore -\alpha\beta^2 = 54$ but $\beta^2 = -\alpha^2$ $\therefore \alpha^3 = 54$ $\alpha = \sqrt[3]{54} \in \mathbb{R}$ $\therefore a = -\alpha = -\sqrt[3]{54} = -3\sqrt[3]{2} = -(54)^{\frac{1}{3}}$ $s_2 = \Sigma\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = b$ $\alpha(\beta+\gamma) + \beta\gamma = b$ $\Rightarrow \beta(-\beta) = b$ $-\beta^2 = b$ $+\alpha^2 = b$ $\therefore b = (-\sqrt[3]{54}) = \sqrt[3]{54^2} = 54^{\frac{2}{3}}$ So $a = -\sqrt[3]{54}$ and $b = \sqrt[3]{54^2}$		$\boxed{2}$

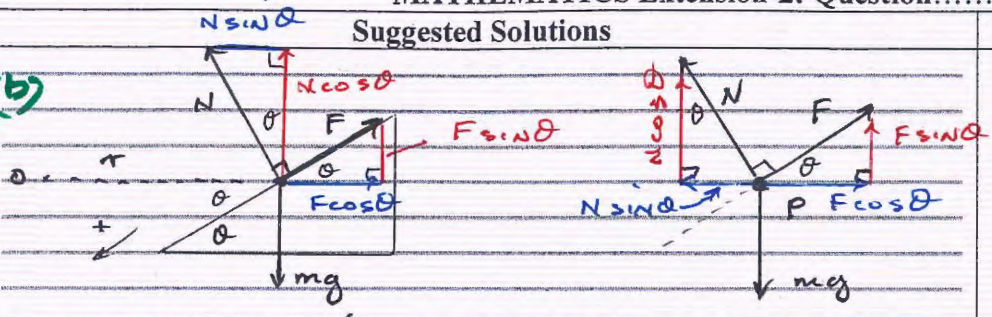
$54^2 = 2916$

JRAHS M. EXT 2 TRIAL, 2012
SECTION II SOLUTIONS

MATHEMATICS Extension 2: Question 15

Suggested Solutions	Marks	Marker's Comments

(b)



(i) Given vertical resolution at P is

$N\cos\theta + F\sin\theta = mg$

HORIZONTAL is $N\sin\theta - F\cos\theta = \dfrac{mv^2}{r}$ (Newton's II) $\boxed{1}$

(ii) $r = 80$, $\angle\theta = 45^\circ$, $g = 10$ $\quad F \le \dfrac{N}{9}$

So min. speed when $F = \dfrac{N}{9}$ or $N = 9F$

$\therefore \dfrac{9F}{\sqrt{2}} + \dfrac{F}{\sqrt{2}} = m \times 10$

$F = m\sqrt{2}$ and

$\dfrac{9F}{\sqrt{2}} - \dfrac{F}{\sqrt{2}} = \dfrac{mv^2}{80}$

$\therefore mv^2 = \dfrac{800F}{\sqrt{2}} = 640m$

$\therefore v^2 = 640$

$v = 8\sqrt{10}$

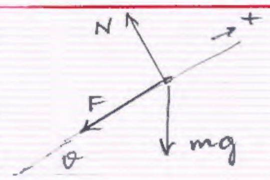
min speed is $8\sqrt{10}$ m/s [91.1 kmph]

$\dfrac{8F}{\sqrt{2}} = \dfrac{mv^2}{80}$

$mv^2 = \dfrac{640F}{\sqrt{2}}$

$\boxed{2}$

(iii) For upwards motion



VERTICAL RESOLUTION: $N\cos\theta - F\sin\theta = mg$

ie $\dfrac{N}{\sqrt{2}} - \dfrac{F}{\sqrt{2}} = 10m$

ie $\dfrac{8F}{\sqrt{2}} = 10m$

ie $F = \dfrac{10m\sqrt{2}}{8}$

HORIZONTAL: $\dfrac{N}{\sqrt{2}} + \dfrac{F}{\sqrt{2}} = \dfrac{mv^2}{80}$

$mv^2 = \dfrac{800F}{\sqrt{2}} = \dfrac{800}{\sqrt{2}} \times \dfrac{10 \times m\sqrt{2}}{8} = 1000m$

$v^2 = 10\,000$

$v = \sqrt{1000} = 10\sqrt{10}$

max speed is $10\sqrt{10}$ m/s [113.8 kmph]

$\dfrac{10F}{\sqrt{2}} = \dfrac{mv^2}{80}$

$\boxed{3}$

(iv) when no slipping $F \equiv 0$ so from (ii) or (iii)

$\dfrac{N}{\sqrt{2}} = 10m$ and $\dfrac{N}{\sqrt{2}} = \dfrac{mV^2}{80}$

$\Rightarrow V^2 = 800$ ie $V = 20\sqrt{2}$ [101.8 kmph]

• can show $V^2 = V_{min} \times V_{max}$ and/or use $\tan\theta = \dfrac{V^2}{rg}$ $\boxed{1}$

11. $\therefore$ the optimum speed for no slippage at 45° is $20\sqrt{2}$ m/s

MATHEMATICS Extension 2: Question 16

Suggested Solutions	Marks	Marker's Comments

(a) $I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$

x	u
1	0
0	1

$x = \frac{1-u}{1+u}$

$\frac{dx}{du} = \frac{-1(1+u)-(1-u)\times 1}{(1+u)^2}$

$= \frac{-1-u-1+u}{(1+u)^2} = \frac{-2}{(1+u)^2}$

$dx = \frac{-2}{(1+u)^2}\,du$

$1+x = 1+\frac{1-u}{1+u} = \frac{2}{1+u}$

$1+x^2 = 1+\frac{(1-u)^2}{(1+u)^2} = \frac{1+2u+u^2+1-2u+u^2}{(1+u)^2}$

$1+x^2 = \frac{2(1+u^2)}{(1+u)^2}$

$\therefore I = \int_1^0 \frac{\ln\frac{2}{1+u} \times \frac{-2}{(1+u)^2}}{\frac{2(1+u^2)}{(1+u)^2}}\,du$

$= \int_1^0 \frac{-2\ln\frac{2}{1+u}}{2(1+u^2)}\,du$

$I = +\int_0^1 \frac{\ln 2}{1+u^2} - \frac{\ln(1+u)}{1+u^2}\,du$

$= \int_0^1 \frac{\ln 2\,dx}{1+x^2} - I$ $\quad u$ is a dummy variable

$2I = \ln 2.\left[\tan^{-1}x\right]_0^1 = \ln 2.\left(\frac{\pi}{4}\right)$

$\therefore I = \ln 2.\frac{\pi}{8}$ q.e.d.

$\boxed{4}$

(b) $x^4+3x-1=0$ METHOD I

so $\alpha^4+3\alpha-1=0$

so $\sum_1^4 \alpha_i^4 + 3\sum_1^4 \alpha_i - \sum_1^4 1 = 0$

$\sum\alpha^4 = -3(\alpha+\beta+\gamma+\delta)+4\times 1 = -3\times 0+4$

$\alpha^4+\beta^4+\gamma^4+\delta^4 = \underline{4}$

Marker's Comments:

Method II

construct Polyn whose roots are $\alpha^4, \beta^4, \gamma^4\ \delta^4$

ie $81x = (1-x)^4$

ie $x^4-4x^3+6x^2-85x+1=0$

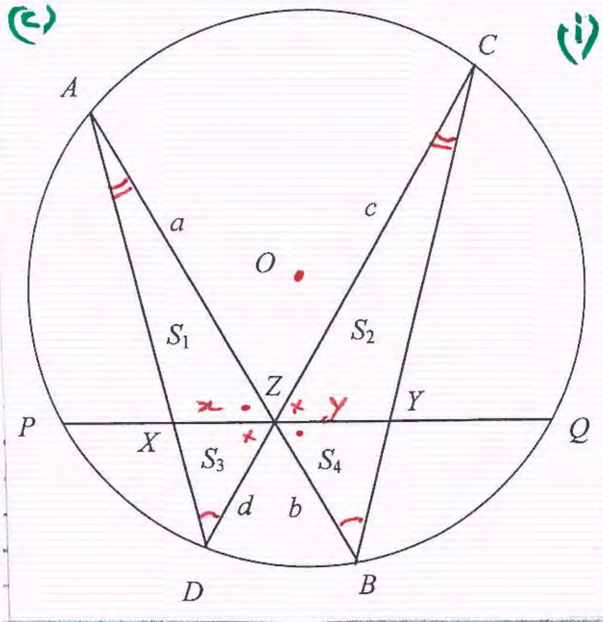
Hence ...

$\boxed{2}$

MATHEMATICS Extension 2: Question 16

Suggested Solutions	Marks	Marker's Comments

(c)



(i) $S_1 = \frac{1}{2} a . AX \sin A$

$S_2 = \frac{1}{2} c . CY . \sin C$

As $\angle C = \angle A$ (angles subtended at circumference are equal standing on arc DB)

$\therefore \sin C = \sin A$

$\therefore \frac{S_1}{S_2} = \frac{a . AX}{c . CY}$ QED

$S_1 = \frac{1}{2} a . x . \sin X\hat{Z}A$

$S_4 = \frac{1}{2} b y \sin B\hat{Z}Y$

but $\angle BZY = \angle XZA$ (vertically opposite angles equal)

$\therefore \frac{S_1}{S_4} = \frac{a . x}{b . y}$ QED

could use ||| Δs

$\boxed{3}$

(ii) similarly $\angle D = \angle B$ (angles subtended at circumference are equal standing on arc AC)

so $\frac{S_3}{S_4} = \frac{\frac{1}{2} DX . d . \sin D}{\frac{1}{2} BY . b . \sin B} = \frac{d . XD}{b . YB}$ ✓

so $\frac{S_1}{S_2} \times \frac{S_3}{S_4} = \frac{a . AX}{c . CY} \times \frac{d . XD}{b . YB} = \frac{a . d \times AX . XD}{b . c \times CY . YB}$

$\boxed{2}$

Now $\frac{S_3}{S_2} = \frac{\frac{1}{2} . d . x . \sin D\hat{Z}X}{\frac{1}{2} . c . y . \sin C\hat{Z}Y} = \frac{d . x}{c . y}$ ✓

$\therefore \frac{S_1}{S_4} \times \frac{S_3}{S_2} = \frac{a . x}{b . y} \times \frac{d . x}{c . y} = \frac{a . d . x^2}{b . c . y^2}$ QED

(iii) $\therefore \frac{a . d \times AX . XD}{b . c \times CY . YB} = \frac{a . d . x^2}{b . c . y^2}$

so $\frac{x^2}{y^2} = \frac{AX . XD}{CY . YB} = \frac{PX . XQ}{PY . YQ} = \frac{(p-x)(x+q)}{(p+y)(q-y)}$

(the products of the intercepts of 2 intersecting chords are equal)

$\boxed{1}$

MATHEMATICS Extension 2: Question 16

Suggested Solutions	Marks	Marker's Comments
(c) (iv) $\dfrac{x^2}{y^2} = \dfrac{(p-x)(x+q)}{(p+y)(q-y)}$		
$= \dfrac{px + pq - x^2 - qx}{pq - py + qy - y^2}$		
$x^2pq - x^2yp + x^2yq - \cancel{x^2y^2} = xy^2p + y^2pq - \cancel{x^2y^2} - xy^2q$		
$pq(x^2 - y^2) = x^2yp - x^2yq + xy^2p - xy^2q$		
$pq(x-y)(x+y) = x^2y(p-q) + xy^2(p-q)$		
$pq(x-y)(x+y) = xy(p-q)(x+y)$		
so $pq(x-y) = xy(p-q)$		
ie $\dfrac{x-y}{xy} = \dfrac{p-q}{pq}$		2
$\therefore \dfrac{1}{y} - \dfrac{1}{x} = \dfrac{1}{q} - \dfrac{1}{p}$ $\Rightarrow \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{p} - \dfrac{1}{q}$		
(v) If Z is the mid point		
$\therefore p = q$		
$\therefore \dfrac{1}{p} - \dfrac{1}{q} = 0$		
$\Rightarrow \underline{x = y}$.		1